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RIGIDITY OF LOCALLY HOMOGENEOUS METRICS OF NEGATIVE CURVATURE ON THE LEAVES OF A FOLIATION

BY

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ABSTRACT

We consider the relationship between diffeomorphism and leafwise isometry for foliations whose leaves are locally homogeneous Riemannian manifolds of negative curvature.

We are concerned with the following problem. Let M be a compact, smooth manifold. Let $\mathcal{F}$, $\mathcal{F}'$ be foliations on M, endowed with Riemannian metrics ω, ω' on the leaves. We assume the metrics are smooth along leaves, and vary continuously in the transverse direction. Assume $\mathcal{F}$, $\mathcal{F}'$ are C^1-equivalent, i.e. that there is a C^1-diffeomorphism of M carrying leaves of $\mathcal{F}$ to leaves of $\mathcal{F}'$. When does this imply that $\mathcal{F}$ and $\mathcal{F}'$ are metrically equivalent, i.e. that there is a homeomorphism of M carrying leaves of $\mathcal{F}$ to those of $\mathcal{F}'$ smoothly, in such a way that ω is carried to ω'? We shall denote the above conclusion by $(*)$.

THEOREM 1. *If the leaves of $\mathcal{F}$ and $\mathcal{F}'$ are locally isometric to a quaternionic hyperbolic space of dimension at least* 8, *or to the Cayley hyperbolic plane, then* $(*)$ *is true.*

THEOREM 2. *If the leaves of $\mathcal{F}$ and $\mathcal{F}'$ are locally isometric to a real or complex hyperbolic space of dimension at least* 3, *and $\mathcal{F}$ admits a holonomy invariant transverse measure which is finite on compact sets and positive on open sets, then* $(*)$ *is true.*

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We remark that if the foliation has just one leaf, then the above assertions reduce to the Mostow rigidity theorem for rank 1. It would be of interest to extend these results to foliations with leaves that are locally symmetric and of higher rank. In this case, under the hypotheses of a transverse measure as in Theorem 2, the existence of a measurable automorphism of M carrying leaves of $\mathscr{F}$ to leaves of $\mathscr{F}'$ smoothly and isometrically follows from the results in [8]. In the case of real hyperbolic space, a weaker version of Theorem 2 was proven in [7], which also asserts the existence of a measurable automorphism of M carrying leaves of $\mathscr{F}$ to leaves of $\mathscr{F}'$ smoothly and isometrically.

The techniques of proof in Theorems 1 and 2 also yield information concerning orbit equivalence for actions of a class of solvable groups introduced in [6].

We recall a definition.

DEFINITION [6]. Let n be a nilpotent Lie algebra, and let $n_1 = n$, $n_{i+1} = [n, n_i]$. By a "good gradation" of n we mean a choice of subspaces $V_i \subset n_i$ such that:

(a) $n_i = n_{i+1} \oplus V_i$,

(b) $[V_i, V_j] \subset V_{i+j}$,

(c) $[V_1, V_i] = V_{i+1}$.

To each good gradation there is a canonical one parameter group of automorphisms $t \to A_t$, where A_t is multiplication by t^i on V_i. We say that a solvable Lie group S is of class (C) if it is the semi-direct product of a simply connected nilpotent Lie group whose Lie algebra has a good gradation, by its canonical one-parameter automorphism group, and the centralizer of A in the automorphism group of N is A itself.

THEOREM 3. *Let S, S' be solvable Lie groups of class* (C). *Suppose we are given locally free C^1-actions of S and S' on a compact manifold. If the actions are C^1-orbit equivalent, then the groups are isomorphic. If in addition the actions are minimal* (*i.e. all orbits are dense*), *then the actions are C^0-conjugate.*

We remark that this result is in sharp contrast to the measurable orbit equivalence theory for actions of these groups [8].

PROOF OF THEOREM 1. We shall first convert the foliation into a group action. Assume a compact manifold M carries of foliation $\mathscr{F}$ with a metric on the leaves such that all leaves are locally isometric to a homogeneous space X. Fix an origin $x_0 \in X$. Let G be the isometry group of X and K the isotropy group of x_0. Define a principal K-bundle P over M in the following way. Let P be the space of continuous maps $p: X \to M$ which are an isometric covering onto a leaf. The projection map $\mu: P \to M$ is defined by $\mu(p) = p(x_0)$. The space P is topologized as a subspace of $C^0(X, M)$. This is sufficient for our purposes but we remark that if $\mathcal{F}$ is smooth, P is a smooth manifold and μ a smooth bundle projection. The group G acts on P by $(g \cdot p)(x) = p(g^{-1}x)$. This action is locally free and its orbits project exactly onto the leaves of $\mathcal{F}$. Restricted to K, the action is simply transitive on each fibre of μ, and turns μ into a principal K-bundle.

Next, assuming that the leaves carry strictly negative curvature, we construct the bundle of asymptotic boundaries of the leaves. Denote by $\bar{X}$ the compactification of X in the sense of the theory of manifolds of negative curvature (see [1]), i.e., $\bar{X}$ is a topological space homeomorphic to a closed disk, and X is embedded in $\bar{X}$ as the interior of the disk. The only property of this construction we shall need is the following: any bi-Lipschitz self map of X extends continuously to $\bar{X}$ (see [2], [3]). Of course, this includes the isometries of X. Define the spaces $E = (X \times P)/K$, $\tilde{E} = (\bar{X} \times P)/K$, where K acts diagonally.

The map $q: E \to M$, $q(x, p) = \mu(p) = p(x_0)$ is a fibration with fibers homeomorpic to X. There is a natural section $\alpha: M \to E$, $m \to$ (x_0, any $p \in P$ such that $p(x_0) = m$). The map $r: E \to M$, $r(x, p) = p(x)$ is a submersion. For each point $m \in M$, the map r is an isometric covering of the fiber E_m on to the leaf at m. In some sense, E is the collection of the universal coverings of leaves of $\mathcal{F}$, the covering map being r.

Let d be a C^1 diffeomorphism of M which sends leaves of $\mathcal{F}$ to leaves of $\mathcal{F}'$. Then, for each leaf of $\mathcal{F}$, the map d is a bi-Lipschitz diffeomorphism of L onto a leaf L'. Indeed, the pulled-back metric $d^*(\omega')$ is continuous, thus there exists a constant Q such that $Q^{-2}\omega \leqq d^*(\omega') \leqq Q^2\omega$. Thus d is bi-Lipschitz with a constant Q^2 which does not depend on the particular leaf L. Let E', $\tilde{E}'$ be the spaces constructed above for the foliation $\mathcal{F}'$. We claim that d extends uniquely to a homeomorphism D between E and E' such that

(i) $q'D = dq$,

(ii) $r'D = dr$.

Indeed, condition (i) means that D maps a fiber E_m onto $E'_{d(m)}$. Now $r: E_m \to L_m$ and $r': E'_{d(m)} \to L'_{d(m)}$ are universal coverings of the leaves. Condition (ii) means that $D: E_m \to E'_{d(m)}$ is nothing but the unique lift of d to the universal covering spaces of L_m and $L'_{d(m)}$ sending $\alpha(m)$ to $\alpha'(d(m))$. Clearly, the map $D : E_m \to E'_{d(m)}$ is bi-Lipschitz, so it extends by continuity to $\tilde{E}_m$. The fact that this extension is continuous on the whole of $\tilde{E}$ follows from the following lemma.

Lemma 4. *Let Y be a topological space, let $D : X \times Y \to X$ be continuous, and bi-Lipschitz on $X \times \{y\}$ with a uniform constant Q. Then the extension of D to $\bar{X} \times Y$ is continuous.*

Proof of Lemma 4. This follows from any proof of the continuous extension of a single bi-Lipschitz map to the asymptotic boundary. See [6] for example.

Now we use results in [6]. Assume that X is a quaternionic hyperbolic space of dimension at least 8 or the Cayley hyperbolic plane. Then any quasi-isometry of X is asymptotic (i.e., has the same boundary extension) to a unique isometry. Thus one obtains a new homeomorphism $F : \tilde{E} \to \tilde{E}'$ such that

(i) $F = D$ on the boundary i.e., on $\tilde{E} - E$; and

(ii) F is an isometry on each fiber of r.

We claim that F can be used to straighten the original diffeomorphism d to a new homeomorphism e of M which is an isometry on the leaves. Indeed, for each $m \in M$, define $e(m) = r'F(\alpha(m))$. Denote by f_m the unique isometry of the leaf L_m onto $L'_{d(m)}$ such that $f_m(m) = e(m)$ and the following diagram commutes:

$$\begin{array}{ccc} E_m & \xrightarrow{F} & E'_{d(m)} \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} \\ L_m & \xrightarrow{f_m} & L'_{d(m)} \end{array}$$

If the leaf L_m has a fundamental group $\Gamma \subset \mathrm{Isom}(E_m)$, then Γ is also the fundamental group of $L'_{d(m)}$, and the map $D : E_m \to E'_{d(m)}$ is Γ-equivariant. The map F is as well and thus f_m is well-defined. We claim that, as the point m varies on a fixed leaf, the map f_m does not vary, i.e., $f_m \equiv e$ for any m. This will prove that e is an isometry, and finishes the proof. Thus, fix m, n in the same leaf L. Choose an isometry $c : E_n \to E_m$ such that $r_m c = r_n$. Choose c' similarly. Then $c'F_n c^{-1} : E_m \to E'_m$ covers f_n with respect to the coverings r_m, r'_m. The maps c, c', being isometries, extend to the boundary, and we continue to denote these extensions by c, c'. The map $c'D_n c^{-1} : E_m \to E'_m$ covers the map $d : L \to L'$ with respect to r_m, r'_m. Hence $c'D_n c^{-1} = \gamma D_m$ for some $\gamma \in \pi_1(L)$

acting on E'_m. This equation is also true on the boundary, and hence $c'F_nc^{-1} = \gamma F_m$. Thus, $c'F_nc^{-1}$ also covers f_m with respect to r_m, r'_m, and hence $f_m = f_n$.

PROOF OF THEOREM 3. The result in [6] is that for the class of solvable groups we consider, quasi-isometric groups are isomorphic and any quasi-isometry is asymptotic to a left translation. In particular, the isometry group of S is S itself. The argument above provides us with an orbit equivalence e of $\mathcal{F}$ to $\mathcal{F}'$ which is an isometry on each leaf. One just needs to check that e conjugates the S actions. For $s \in S$, the map $e^{-1}se$ is a homeomorphism of M preserving each leaf of the foliation $\mathcal{F}$, and which is an isometry of this leaf. For each point $m \in M$, there is a unique element $\sigma(s, m)$ of S such that $\sigma(s, m)m = e^{-1}se(m)$. The map $m \to \sigma(s, m)$ is continuous, and thus constant if we assume that the foliation $\mathcal{F}$ is minimal. Obviously $s \to \sigma(s)$ is a group homomorphism. Interchanging $\mathcal{F}$ and $\mathcal{F}'$ in this argument, we see that σ is an automorphism. Finally we observe that the definition of class (C) implies that every automorphism of S is inner, and hence e can be modified to conjugate the S actions.

PROOF OF THEOREM 2. In the case of foliations by real or complex hyperbolic spaces, the bi-Lipschitz map $D : E \to E'$ is not necessarily asymptotic to an isometry. We can prove this under the extra assumption that $\mathcal{F}$ admits a suitable finite invariant transverse measure. One extends Mostow's argument in [4] and [5]: over each ergodic component of the action of G on P, one shows that the boundary extension of D on $P \times X(\infty)$, which *a priori* is Q-quasiconformal, is in fact 1-quasiconformal, and thus asymptotic to an isometry. This is done in [7] for real hyperbolic space. The argument in the complex case is similar, but we present the arguments here for the sake of completeness.

The diffeomorphism d extends to a measurable orbit equivalence $\theta : P \to P'$ which is a quasi-isometry on the orbits of G. In the same way as we constructed the space E and the maps r and D, we consider the space $G \times P$; the map $r : E \to M$ is replaced by the group action $R : G \times P \to P$ and $D : E \to E'$ is replaced by the cocycle Ω of the action of G:

$$\Omega(g, s) = (g', \theta(s)) \qquad \text{where } \theta(gs) = g'\theta(s).$$

On each factor $G \times \{s\}$, the quasi-isometry $\Omega_s : G \times \{s\} \to G \times \{\theta(s)\}$ extends to a quasiconformal homeomorphism of the boundary sphere $X(\infty)$. In the sequel, we use the Poincaré disk model, where real (resp. complex) hyperbolic n-space is seen as a ball in Euclidean (resp. Hermitian) n-space.

The boundary sphere carries an induced Riemannian (resp. Carnot–Caratheodory) metric for which the group G acts conformally. Recall that the round sphere S^{2n-1} in $\mathbf{C}^n$ carries a canonical contact structure: the contact hyperplane W_x at $x \in S^{2n-1}$ is the union of all complex lines in $T_x S^{2n-1}$. The Carnot–Caratheodory metric is a metric on this hyperplane field. Clearly, if $s \in P$ projects onto $m \in M$, then Ω coincides with D modulo conformal mappings identifying $X(\infty)$ with $\tilde{E}_m$ and $\tilde{E}'_{d(m)}$. Thus we need only show that Ω is 1-quasiconformal on the boundary. That is, at each point where Ω is differentiable, we show that the differential is a similitude. (In the complex case, the differential should be understood in the sense of [6].)

Let $j: RX(\infty) \to X(\infty)$ be the bundle of tangent rays to $X(\infty)$ in the real case, or the bundle of rays lying in the contact hyperplanes W_x, $x \in X(\infty)$, in the complex case. Then $RX(\infty)$ is a compact space on which G acts transitively. Let ν be the product measure on $X(\infty) \times P$, so that ν is quasi-invariant under G. If $Y \subset X(\infty) \times P$ is (almost) any ν-ergodic component, then G is still ergodic on $\tilde{Y} = j^{-1}(Y) \times P$. To see this, we recall that by Moore's theorem (see [8]), any cocompact subgroup of G acts ergodically on an ergodic G-space with a finite invariant measure, or equivalently, that for any cocompact $G_1 \subset G$, G acts ergodically on the product of G/G_1 with any ergodic G-space of finite invariant measure. It follows that Y is itself the product of $X(\infty)$ with an ergodic component of the action on P (which we recall has finite invariant measure.) Ergodicity on $\tilde{Y}$ follows since G is transitive on $RX(\infty)$ with non-compact stabilizer.

We note that a linear map λ between Euclidean spaces is a similitude if and only if the ratio $|\lambda(v)|/|v|$ is constant. Therefore, non-conformality can be measured by a single function on $RX(\infty)$. Namely, at each point $(x, s) \in X(\infty) \times P$ where Ω is differentiable, we define for $r \in R_x X(\infty)$, $\phi(r, s) = |d\Omega(v)|/|d\Omega|\,|v|$ for some $v \in r$. We claim that ϕ is G-invariant. Indeed, if λ, λ' are similitudes,

$$|\lambda \circ d\Omega \circ \lambda'(v)|/|\lambda \circ d\Omega \circ \lambda'|\,|v| = |d\Omega(\lambda' v)|/|d\Omega|\,|\lambda'(v)|.$$

From the cocycle equation $\Omega_{gs}(h)\Omega_s(g) = \Omega_s(hg)$ where $s \in P$, $g, h \in G$, we obtain

$$\Omega_{gs} = \beta(g') \circ \Omega_s \circ \beta(g^{-1})$$

where $g' = \Omega_s(g) \in G$, and β denotes the action of G on $X(\infty)$ by conformal mappings. Taking differentials, we conclude $\phi(gr, gs) = \phi(r, s)$. Thus ϕ is constant a.e. on $\tilde{Y}$, for (almost) every ergodic component Y. It follows that almost every boundary extension Ω_s is 1-quasi-conformal on the boundary, and hence conformal [5], [6]. It follows from Lemma 4 that every boundary extension is conformal, and hence the boundary extension of a unique isometry. The proof is then completed as in the proof of Theorem 1.

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